

The (2+1)-dimensional Hirota-Maxwell-Bloch equation: Darboux transformation and soliton solutions

Kuralay Yesmakhanova, Gaukhar Shaikhova, Kuanysh Zhussupbekov
and Ratbay Myrzakulov*

*Eurasian International Center for Theoretical Physics and Department of General
& Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan*

Abstract

The (2+1)-dimensional Hirota-Maxwell-Bloch equation (HMBE) is integrable by the Inverse Scattering Method. In this paper, we construct a Darboux transformation (DT) of the (2+1)-dimensional HMBE. Also the one-soliton solution obtained by means of the one-fold DT. For the n -soliton solution the general form is presented.

1 Introduction

Modern nonlinear science as a powerful subject explains all kinds of mysteries in the challenges of modern technology and science. The nonlinear nature of the real systems is considered to be fundamental to the understanding of most natural phenomena. Integrable systems are the main part of theory of modern nonlinear science. One of the interesting integrable system is the so-called (1+1)-dimensional Hirota-Maxwell-Bloch equation. It describes the nonlinear dynamics of femtosecond pulse propagation through doped fibre. In this paper, we consider one of (2+1)-dimensional integrable generalizations of the (1+1)-dimensional HMBE, namely, the (2+1)-dimensional HMBE. We present the Darboux transformation and using it, the one-soliton solution.

The paper is organized as follows. In Section 2, we give a brief review of the (1+1)-dimensional HMBE. The (2+1)-dimensional HMBE we present in Section 3. In Section 4, we construct the DT for the (2+1)-dimensional HMBE. In Section 5, using the constructed one-fold DT, the one-soliton solution of the (2+1)-dimensional HMBE is given. In Section 6, conclusions are given.

2 Brief review of the (1+1)-dimensional Hirota -Maxwell-Bloch equation

To establish our notation, definitions and terminology let us first recall some main informations on the (1+1)-dimensional Hirota -Maxwell-Bloch equation (HMBE). The (1+1)-dimensional HMBE has the form (see e.g. [1]-[3], [4]-[8])

$$iq_t + \epsilon_1(q_{xx} + 2\delta|q|^2q) + i\epsilon_2(q_{xxx} + 6\delta|q|^2q_x) - 2ip = 0, \quad (2.1)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (2.2)$$

$$\eta_x + \delta(q^*p + p^*q) = 0, \quad (2.3)$$

where q, p are complex functions, η is a real function and ω, ϵ_i are real constants. By setting $\delta = +1$ or $\delta = -1$, the HMBE with attractive or repulsive interaction is obtained. The functions p and η satisfy the relation

$$\eta^2 + \delta|p|^2 = b(t) = \text{const}(t). \quad (2.4)$$

*The corresponding author. Email: rmyrzakulov@gmail.com

The DT and some exact solutions of this equation were presented in [1]-[3]. Note that the gauge [4] and Lakshmanan equivalent counterpart [7] of the (1+1)-dimensional HMBE (2.1)-(2.3) is the following Myrzakulov-XCIV equation (M-XCIV equation) [4]:

$$iS_t + \frac{1}{2}\epsilon_1[S, S_{xx}] + i\epsilon_2(S_{xx} + 6tr(S_x^2)S)_x + \frac{1}{\omega}[S, W] = 0, \quad (2.5)$$

$$iW_x + \omega[S, W] = 0, \quad (2.6)$$

where $S = S_i\sigma_i$, $W = W_i\sigma_i$, $S^2 = I$, $W^2 = b(t)I$, $b(t) = \text{const}(t)$, $I = \text{diag}(1, 1)$, $[A, B] = AB - BA$, ω is a real constant and σ_i are Pauli matrices. The DT of the M-XCIV equation and its one-soliton solution were constructed in [8].

3 The (2+1)-dimensional Hirota -Maxwell-Bloch equation

The (2+1)-dimensional Hirota -Maxwell-Bloch equation (HMBE) reads as [5]

$$iq_t + \epsilon_1 q_{xy} + i\epsilon_2 q_{xxy} - vq + i(wq)_x - 2ip = 0, \quad (3.1)$$

$$v_x + 2\epsilon_1\delta(|q|^2)_y - 2i\epsilon_2\delta(q_{xy}^*q - q^*q_{xy}) = 0, \quad (3.2)$$

$$w_x - 2\epsilon_2\delta(|q|^2)_y = 0, \quad (3.3)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (3.4)$$

$$\eta_x + \delta(q^*p + p^*q) = 0, \quad (3.5)$$

where q, p are complex functions, v, w, η are real functions. This set of equations (3.1)-(3.5) is integrable by IST. The corresponding Lax representation reads as

$$\Psi_x = A\Psi, \quad (3.6)$$

$$\Psi_t = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\Psi_y + B\Psi, \quad (3.7)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (3.8)$$

$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}. \quad (3.9)$$

Here

$$B_1 = iw\sigma_3 + 2i\epsilon_2\sigma_3A_{0y}, \quad (3.10)$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (3.11)$$

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & i\epsilon_1q_y - \epsilon_2q_{xy} - wq \\ i\epsilon_1r_y + \epsilon_2r_{xy} + wr & 0 \end{pmatrix}, \quad (3.12)$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix} \quad (3.13)$$

and $r = \delta q^*$, $k = \delta p^*$, where $\delta = \pm 1$. The spectral parameter λ evolves as

$$\lambda_t = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\lambda_y. \quad (3.14)$$

In this paper we restrict ourselves to the case $\delta = +1$ that is to the focusing (attractive interaction) case. We note that if $y = x$ the system (3.1)-(3.5) reduces to the (1+1)-dimensional HMBE (2.1)-(2.3). This fact explains why we called the system (3.1)-(3.5) as the (2+1)-dimensional HMBE. At last, we present the Myrzakulov-Lakshmanan-IV equation (ML-IV equation) [5]

$$iS_t + 2\epsilon_1Z_x + i\epsilon_2(S_{xy} + [S_x, Z])_x + (fS)_x + \frac{1}{\omega}[S, W] = 0, \quad (3.15)$$

$$u_x - \frac{i}{4}tr(S \cdot [S_x, S_y]) = 0, \quad (3.16)$$

$$f_x - \frac{i}{4}\epsilon_2[tr(S_x^2)]_y = 0, \quad (3.17)$$

$$iW_x + \omega[S, W] = 0, \quad (3.18)$$

where

$$Z = 0.5([S, S_y + 2iuS]). \quad (3.19)$$

As it was shown in [5], the ML-IV equation is the equivalent counterpart of the (2+1)-dimensional HMBE. In 1+1 dimensions, the ML-IV equation reduces to the M-XCIV equation (2.5)-(2.6).

4 DT for the (2+1)-dimensional HMBE

In this section we construct the DT for the (2+1)-dimensional HMBE (3.14)-(3.17). In particular, we give in detail the one-fold DT and briefly the n -fold DT.

4.1 One-fold DT

Let Ψ and Ψ' are two solutions of the system (3.6)-(3.7) so that

$$\Psi'_x = A'\Psi', \quad (4.1)$$

$$\Psi'_t = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\Psi'_y + B'\Psi'. \quad (4.2)$$

We assume that these two solutions are related by the following transformation:

$$\Psi' = T\Psi = (\lambda I - M)\Psi. \quad (4.3)$$

The matrix function T obeys the following equations

$$T_x + TA = A'T, \quad (4.4)$$

$$T_t + TB = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)T_y + B'T. \quad (4.5)$$

From the equation (4.4) we get

$$\lambda^0 : M_x = A'_0 M - M A_0, \quad (4.6)$$

$$\lambda^1 : A'_0 = A_0 + i[M, \sigma_3], \quad (4.7)$$

$$\lambda^2 : iI\sigma_3 = i\sigma_3 I. \quad (4.8)$$

Eq.(4.7) gives

$$q^{[1]} = q - 2im_{12}, \quad (4.9)$$

$$q^{*[1]} = q^* - 2im_{21}, \quad (4.10)$$

where

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.11)$$

Hence we get $m_{21} = -m_{12}^*$ in our attractive interaction case. Eq.(4.5) gives us the following relations

$$I : \lambda_t = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\lambda_y, \quad (4.12)$$

$$\lambda^0 : -M_t = iB'_{-1} - B'_0 M - iB_{-1} + MB_0, \quad (4.13)$$

$$\lambda^1 : 2\epsilon_1 M_y = B'_0 - B_0 + MB_1 - B'_1 M, \quad (4.14)$$

$$\lambda^2 : 4\epsilon_2 M_y = B'_1 - B_1, \quad (4.15)$$

$$(\lambda + \omega)^{-1} : 0 = -i\omega B'_{-1} - iB'_{-1} M + i\omega B_{-1} + iMB_{-1}. \quad (4.16)$$

Hence we get the DT

$$B'_0 = B_0 - MB_1 + (B_1 + 4\epsilon_2 M_y)M + 2\epsilon_1 M_y, \quad (4.17)$$

$$B'_1 = B_1 + 4\epsilon_2 M_y, \quad (4.18)$$

$$B'_{-1} = (M + \omega I)B_{-1}(M + \omega I)^{-1}. \quad (4.19)$$

At the same, from Eqs.(4.17)-(4.18) we get

$$v' = v + 4i\epsilon_1 m_{11y} + 4m_{11}w + 4\epsilon_2(m_{12}q_y^* - m_{12}^*q_y + 2im_{11}m_{11y} - 2im_{12}^*m_{12y}), \quad (4.20)$$

$$w' = w - 4i\epsilon_2 m_{11y} = w + 4i\epsilon_2 m_{22y} \quad (4.21)$$

and we additionally have $m_{22} = m_{11}^*$. So the matix M has the form

$$M = \begin{pmatrix} m_{11} & m_{12} \\ -m_{12}^* & m_{11}^* \end{pmatrix}, \quad M^{-1} = \frac{1}{|m_{11}|^2 + |m_{12}|^2} \begin{pmatrix} m_{11}^* & -m_{12} \\ m_{12}^* & m_{11} \end{pmatrix}, \quad (4.22)$$

$$M + \omega I = \begin{pmatrix} m_{11} + \omega & m_{12} \\ -m_{12}^* & \omega + m_{11}^* \end{pmatrix}, \quad (M + \omega I)^{-1} = \frac{1}{\square} \begin{pmatrix} m_{11}^* + \omega & -m_{12} \\ m_{12}^* & \omega + m_{11} \end{pmatrix}, \quad (4.23)$$

where

$$\square = \det(M + \omega I) = \omega^2 + \omega(m_{11} + m_{11}^*) + |m_{11}|^2 + |m_{12}|^2. \quad (4.24)$$

The equation (4.19) gives

$$\eta' = \frac{(|\omega + m_{11}|^2 - |m_{12}|^2)\eta - pm_{12}^*(\omega + m_{11}) - p^*m_{12}(\omega + m_{11}^*)}{\square}, \quad (4.25)$$

$$p' = \frac{p(\omega + m_{11})^2 - p^*m_{12}^2 + 2\eta m_{12}(\omega + m_{11})}{\square}, \quad (4.26)$$

$$p^{*'} = \frac{p^*(\omega + m_{11}^*)^2 - pm_{12}^{*2} + 2\eta m_{12}^*(\omega + m_{11}^*)}{\square}. \quad (4.27)$$

We now assume that

$$M = H\Lambda H^{-1}, \quad (4.28)$$

where

$$H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & \psi_1(\lambda_2; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_2(\lambda_2; t, x, y) \end{pmatrix}. \quad (4.29)$$

Here

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (4.30)$$

and $\det H \neq 0$, where λ_1 and λ_2 are complex constants. The matrix H obeys the system

$$H_x = -i\sigma_3 H \Lambda + A_0 H, \quad (4.31)$$

$$H_t = 2H_y \Lambda + B_0 H + B_{-1} H \Sigma, \quad (4.32)$$

where

$$\Sigma = \begin{pmatrix} \frac{i}{\lambda_1 + \omega} & 0 \\ 0 & \frac{i}{\lambda_2 + \omega} \end{pmatrix}. \quad (4.33)$$

In order to satisfy the constraints of S and B'_{-1} as mentioned above, we first notes that

$$\Psi^+ = \Psi^{-1}, \quad A_0^+ = -A_0, \quad (4.34)$$

$$\lambda_2 = \lambda_1^*, \quad H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & -\psi_2^*(\lambda_1; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_1^*(\lambda_1; t, x, y) \end{pmatrix}, \quad (4.35)$$

$$H^{-1} = \frac{1}{\Delta} \begin{pmatrix} \psi_1^*(\lambda_1; t, x, y) & \psi_2^*(\lambda_1; t, x, y) \\ -\psi_2(\lambda_1; t, x, y) & \psi_1(\lambda_1; t, x, y) \end{pmatrix}, \quad (4.36)$$

where

$$\Delta = |\psi_1|^2 + |\psi_2|^2. \quad (4.37)$$

So the matrix M has the form

$$M = \frac{1}{\Delta} \begin{pmatrix} \lambda_1 |\psi_1|^2 + \lambda_2 |\psi_2|^2 & (\lambda_1 - \lambda_2) \psi_1 \psi_2^* \\ (\lambda_1 - \lambda_2) \psi_1^* \psi_2 & \lambda_1 |\psi_2|^2 + \lambda_2 |\psi_1|^2 \end{pmatrix}. \quad (4.38)$$

Finally we can write the one-fold DT for the (2+1)-dimensional HMBE as:

$$q^{[1]} = q - 2im_{12}, \quad (4.39)$$

$$v^{[1]} = v + 4i\epsilon_1 m_{11y} + 4m_{11}w + 4\epsilon_2(m_{12}q_y^* - m_{12}^*q_y - 2im_{11}m_{11y} + 4im_{12}^*m_{12y}), \quad (4.40)$$

$$w^{[1]} = w - 4i\epsilon_2 m_{11y} = w + 4i\epsilon_2 m_{22y}, \quad (4.41)$$

$$\eta^{[1]} = \frac{(|\omega + m_{11}|^2 - |m_{12}|^2)\eta - pm_{12}^*(\omega + m_{11}) - p^*m_{12}(\omega + m_{11}^*)}{\square}, \quad (4.42)$$

$$p^{[1]} = \frac{p(\omega + m_{11})^2 - p^*m_{12}^2 + 2\eta m_{12}(\omega + m_{11})}{\square}. \quad (4.43)$$

At last, we note that the expressions of m_{ij} can be rewritten in the determinant form as

$$m_{11} = \frac{\lambda_1 |\psi_1|^2 + \lambda_2 |\psi_2|^2}{\Delta} = \frac{\Delta_{11}}{\Delta}, \quad m_{12} = \frac{(\lambda_1 - \lambda_2) \psi_1 \psi_2^*}{\Delta} = \frac{\Delta_{12}}{\Delta}, \quad (4.44)$$

where

$$\Delta_{11} = \det \begin{pmatrix} \psi_1 & -\lambda_2 \psi_2^* \\ \psi_2 & \lambda_1 \psi_1^* \end{pmatrix}, \quad \Delta_{12} = -\det \begin{pmatrix} \psi_1 & \lambda_1 \psi_1 \\ \psi_2^* & \lambda_2 \psi_2^* \end{pmatrix}. \quad (4.45)$$

4.2 n -fold DT

In the previous subsection we have constructed the one-fold DT. Similarly we can construct the n -fold DT. To construct the n -fold DT of the (2+1)-dimensional HMBE, we note that the function T satisfies the following equations

$$T_{nx} = A^{[n]}T_n - T_n A, \quad (4.46)$$

$$T_{nt} = (2\epsilon_1 \lambda + 4\epsilon_2 \lambda^2)T_{ny} + B^{[n]}T_n - T_n B. \quad (4.47)$$

The solution of this system can be written as [1]

$$T_n(\lambda; \lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots, \lambda_{2n}) = \lambda^n I + t_{n-1}^{[n]} \lambda^{n-1} + \dots + t_1^{[n]} \lambda + t_0^{[n]} = \frac{1}{\Delta_n} \begin{pmatrix} T_{n11} & T_{n12} \\ T_{n21} & T_{n22} \end{pmatrix}, \quad (4.48)$$

where [1]

$$\Delta_n = \begin{vmatrix} \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} \end{vmatrix} \quad (4.49)$$

$$T_{n11} = \begin{vmatrix} 1 & 0 & \lambda & 0 & \dots & \lambda^{n-1} & 0 & \lambda^n \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} & \lambda_1^n \Phi_{1,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} & \lambda_2^n \Phi_{1,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} & \lambda_3^n \Phi_{1,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} & \lambda_4^n \Phi_{1,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} & \lambda_{2n-1}^n \Phi_{1,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} & \lambda_{2n}^n \Phi_{1,2n} \end{vmatrix} \quad (4.50)$$

$$T_{n12} = \begin{vmatrix} 0 & 1 & 0 & \lambda & \dots & 0 & \lambda^{n-1} & 0 \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} & \lambda_1^n \Phi_{1,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} & \lambda_2^n \Phi_{1,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} & \lambda_3^n \Phi_{1,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} & \lambda_4^n \Phi_{1,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} & \lambda_{2n-1}^n \Phi_{1,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} & \lambda_{2n}^n \Phi_{1,2n} \end{vmatrix} \quad (4.51)$$

$$T_{n21} = \begin{vmatrix} 1 & 0 & \lambda & 0 & \dots & \lambda^{n-1} & 0 & 0 \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} & \lambda_1^n \Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} & \lambda_2^n \Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} & \lambda_3^n \Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} & \lambda_4^n \Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} & \lambda_{2n-1}^n \Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} & \lambda_{2n}^n \Phi_{2,2n} \end{vmatrix} \quad (4.52)$$

$$T_{n22} = \begin{vmatrix} 0 & 1 & 0 & \lambda & \dots & 0 & \lambda^{n-1} & \lambda^n \\ \Phi_{1,1} & \Phi_{2,1} & \lambda_1 \Phi_{1,1} & \lambda_1 \Phi_{2,1} & \dots & \lambda_1^{n-1} \Phi_{1,1} & \lambda_1^{n-1} \Phi_{2,1} & \lambda_1^n \Phi_{2,1} \\ \Phi_{1,2} & \Phi_{2,2} & \lambda_2 \Phi_{1,2} & \lambda_2 \Phi_{2,2} & \dots & \lambda_2^{n-1} \Phi_{1,2} & \lambda_2^{n-1} \Phi_{2,2} & \lambda_2^n \Phi_{2,2} \\ \Phi_{1,3} & \Phi_{2,3} & \lambda_3 \Phi_{1,3} & \lambda_3 \Phi_{2,3} & \dots & \lambda_3^{n-1} \Phi_{1,3} & \lambda_3^{n-1} \Phi_{2,3} & \lambda_3^n \Phi_{2,3} \\ \Phi_{1,4} & \Phi_{2,4} & \lambda_4 \Phi_{1,4} & \lambda_4 \Phi_{2,4} & \dots & \lambda_4^{n-1} \Phi_{1,4} & \lambda_4^{n-1} \Phi_{2,4} & \lambda_4^n \Phi_{2,4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{1,2n-1} & \Phi_{2,2n-1} & \lambda_{2n-1} \Phi_{1,2n-1} & \lambda_{2n-1} \Phi_{2,2n-1} & \dots & \lambda_{2n-1}^{n-1} \Phi_{1,2n-1} & \lambda_{2n-1}^{n-1} \Phi_{2,2n-1} & \lambda_{2n-1}^n \Phi_{2,2n-1} \\ \Phi_{1,2n} & \Phi_{2,2n} & \lambda_{2n} \Phi_{1,2n} & \lambda_{2n} \Phi_{2,2n} & \dots & \lambda_{2n}^{n-1} \Phi_{1,2n} & \lambda_{2n}^{n-1} \Phi_{2,2n} & \lambda_{2n}^n \Phi_{2,2n} \end{vmatrix} \quad (4.53)$$

The corresponding n -fold DT is given by

$$A_0^{[n]} = A_0 + \frac{i}{(n-1)!} \left[\sigma_3, \frac{\partial^{n-1} T_n}{\partial \lambda^{n-1}} \right], \quad (4.54)$$

$$B_1^{[n]} = B_1 - 4\epsilon_2 (T|_{\lambda=0})_y, \quad (4.55)$$

$$B_{-1}^{[n]} = T_n|_{\lambda=-\omega} B_{-1} T_n^{-1}|_{\lambda=-\omega}. \quad (4.56)$$

5 Soliton solutions

Having the explicit form of the DT, we are ready to construct exact solutions of the (2+1)-dimensional HMBE. As an example, let us present the one-soliton solution. To get the one-soliton solution we take the seed solution as

$$q = v = w = p = 0, \quad \eta = 1. \quad (5.1)$$

Then the corresponding associated linear system takes the form

$$\psi_{1x} = -i\lambda\psi_1, \quad (5.2)$$

$$\psi_{2x} = i\lambda\psi_2, \quad (5.3)$$

$$\psi_{1t} = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\psi_{1y} + \frac{i}{\lambda + \omega}\psi_1, \quad (5.4)$$

$$\psi_{2t} = (2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\psi_{2y} - \frac{i}{\lambda + \omega}\psi_2. \quad (5.5)$$

This system admits the following exact solutions

$$\psi_1 = e^{-i\lambda_1 x + i\mu_1 y + i[(2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\mu_1 + \frac{1}{\lambda_1 + \omega}]t + \delta_1 + i\delta_2}, \quad (5.6)$$

$$\psi_2 = e^{i\lambda_1 x - i\mu_1 y - i[(2\epsilon_1\lambda + 4\epsilon_2\lambda^2)\mu_1 + \frac{1}{\lambda_1 + \omega}]t - \delta_1 - i\delta_2 + i\delta_0} \quad (5.7)$$

or

$$\psi_1 = e^{\theta_1 + i\chi_1}, \quad (5.8)$$

$$\psi_2 = e^{\theta_2 + i\chi_2}. \quad (5.9)$$

Here $\mu_1 = \eta + i\nu$, $\lambda_1 = \alpha + i\beta$, δ_i are real constants,

$$\theta_1 = \beta x - \nu y - \{\eta(2\epsilon_1\beta + 8\epsilon_2\alpha\beta) + \nu[2\epsilon_1\alpha + 4\epsilon_2(\alpha^2 - \beta^2)] + \frac{\beta}{(\alpha + \omega)^2 + \beta^2}\}t + \delta_1 \quad (5.10)$$

$$\chi_1 = -\alpha x + \eta y + \{\eta(2\epsilon_1\alpha + 4\epsilon_2(\alpha^2 - \beta^2)) - \nu[2\epsilon_1\beta + 8\epsilon_2\alpha\beta] + \frac{\alpha + \omega}{(\alpha + \omega)^2 + \beta^2}\}t + \delta_2 \quad (5.11)$$

and $\theta_2 = -\theta_1$, $\chi_2 = -\chi_1 + \delta_0$. Then the one-soliton solution of the (2+1)-dimensional HMBE (3.1)-(3.5) takes the form

$$q^{[1]} = -2im_{12}, \quad (5.12)$$

$$v^{[1]} = 4i\epsilon_1 m_{11y} + 4\epsilon_2(-2im_{11}m_{11y} + 4im_{12}^*m_{12y}), \quad (5.13)$$

$$w^{[1]} = -4i\epsilon_2 m_{11y}, \quad (5.14)$$

$$\eta^{[1]} = \frac{|\omega + m_{11}|^2 - |m_{12}|^2}{\square}, \quad (5.15)$$

$$p^{[1]} = \frac{2m_{12}(\omega + m_{11})}{\square}, \quad (5.16)$$

where

$$m_{11} = \alpha + i\beta \tanh[2\theta_1], \quad (5.17)$$

$$m_{12} = \frac{i\beta e^{2i\chi_1 - i\delta_0}}{\cosh[2\theta_1]}, \quad (5.18)$$

and

$$\square = \omega^2 + 2\alpha\omega + \alpha^2 + \beta^2 = (\omega + \alpha)^2 + \beta^2. \quad (5.19)$$

Finally let us present the one-soliton solutions for some particular cases.

a) If $\epsilon_1 = 1$, $\epsilon_2 = 0$ then the HMBE (3.1)-(3.5) turn to the (2+1)-dimensional Schrödinger-Maxwell-Bloch equation (SMBE) of the form

$$iq_t + q_{xy} - vq - 2ip = 0, \quad (5.20)$$

$$v_x + 2\delta(|q|^2)_y = 0, \quad (5.21)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.22)$$

$$\eta_x + \delta(q^*p + p^*q) = 0, \quad (5.23)$$

where $\delta = 1$ for our case. Its one-soliton solution has the form

$$q^{[1]} = -2im'_{12}, \quad (5.24)$$

$$v^{[1]} = 4im'_{11y}, \quad (5.25)$$

$$\eta^{[1]} = \frac{|\omega + m'_{11}|^2 - |m'_{12}|^2}{\square}, \quad (5.26)$$

$$p^{[1]} = \frac{2m'_{12}(\omega + m'_{11})}{\square}, \quad (5.27)$$

where $m'_{11} = m_{11}|_{\epsilon_1=1, \epsilon_2=0}$ and $m'_{12} = m_{12}|_{\epsilon_1=1, \epsilon_2=0}$. This solution was obtained in [6].

b) Now let we put $\epsilon_1 = 0$, $\epsilon_2 = 1$. In this case, the HMBE (3.1)-(3.5) becomes the complex modified Korteweg-de Vries-Maxwell-Bloch equation (cmKdVMBE) of the form

$$q_t + q_{xxy} + ivq + (wq)_x - 2p = 0, \quad (5.28)$$

$$v_x - 2i\delta(q_{xy}^*q - q^*q_{xy}) = 0, \quad (5.29)$$

$$w_x - 2\delta(|q|^2)_y = 0, \quad (5.30)$$

$$p_x - 2i\omega p - 2\eta q = 0, \quad (5.31)$$

$$\eta_x + \delta(q^*p + p^*q) = 0, \quad (5.32)$$

where $\delta = +1$. The one-soliton solution of the (2+1)-dimensional cmKdVMBE follows from (5.12)-(5.16) and has the form

$$q^{[1]} = -2im''_{12}, \quad (5.33)$$

$$v^{[1]} = 4(-2im''_{11}m''_{11y} + 4im''_{12}m''_{12y}), \quad (5.34)$$

$$w^{[1]} = -4im''_{11y}, \quad (5.35)$$

$$\eta^{[1]} = \frac{|\omega + m''_{11}|^2 - |m''_{12}|^2}{\square}, \quad (5.36)$$

$$p^{[1]} = \frac{2m''_{12}(\omega + m''_{11})}{\square}, \quad (5.37)$$

where $m''_{11} = m_{11}|_{\epsilon_1=0, \epsilon_2=1}$ and $m''_{12} = m_{12}|_{\epsilon_1=0, \epsilon_2=1}$.

6 Conclusion

The DT is very useful to derive all kinds of solutions of integrable equations. Here the DT for the (2+1)-dimensional HMBE is constructed. In particular, the one-fold DT is presented in detail. The general formulas for the n -fold DT is also given. Using the derived one-fold DT, the one-soliton solution of the (2+1)-dimensional HMBE is found. We note that using the above presented DT, one can also construct the n -solitons, breathers, rogue waves and other type exact solutions of the (2+1)-dimensional HMBE. Finally we note that these results can be extend to the other nonlinear equations including integrable spin systems [9]-[33].

References

- [1] Chuanzhong Li, Jingsong He, K. Porsezian. *Rogue waves of the Hirota and the Maxwell-Bloch equations*, [arXiv:1205.1191]
- [2] Chuanzhong Li, Jingsong He. *Darboux transformation and positons of the inhomogeneous Hirota and the Maxwell-Bloch equation*, [arXiv:1210.2501]
- [3] Jieming Yang, Chuanzhong Li, Tiantian Li, Zhaoneng Cheng. *Darboux transformation and solutions of the two-component Hirota-Maxwell-Bloch system*, Chin. Phys Lett., **30**, N10, 104201 (2013). arXiv:1310.0617.
- [4] Zh. Kh. Zhunussova, K. R. Yesmakhanova, D. I. Tungushbaeva, G. K. Mamyrbekova, G. N. Nugmanova, R. Myrzakulov. *Integrable Heisenberg Ferromagnet Equations with self-consistent potentials*. [arXiv:1301.1649]
- [5] R. Myrzakulov, G. K. Mamyrbekova, G. N. Nugmanova, M. Lakshmanan. *Integrable (2+1)-dimensional spin models with self-consistent potentials*. [arXiv:1305.0098]
- [6] G. Shaikhova, K. Yesmakhanova, G. Mamyrbekova, R. Myrzakulov. *Darboux tranformation and solutions of the (2+1)-dimensional Schrodinger-Maxwell-Bloch equation*, [arXiv:1402.4669]
- [7] R. Myrzakulov, G. K. Mamyrbekova, G.N. Nugmanova, K.R. Yesmakhanova, M. Lakshmanan. *Integrable Motion of Curves in Self-Consistent Potentials : Relation to Spin Systems and Soliton Equations*, [arXiv:1404.2088]
- [8] Z.S. Yersultanova, M. Zhassybayeva, G. Mamyrbekova, G. Nugmanova, R. Myrzakulov. *Darboux Transformation and Exact Solutions of the integrable Heisenberg ferromagnetic equation with self-consistent potentials*, [arXiv:1404.2270]
- [9] M. Lakshmanan, Phil. Trans. R. Soc. A, **369** 1280-1300 (2011).
- [10] M. Lakshmanan, Phys. Lett. A **A** 53-54 (1977).
- [11] L.A. Takhtajan, Phys. Lett. A **64** 235-238 (1977).

- [12] C. Senthilkumar, M. Lakshmanan, B. Grammaticos, A. Ramani, Phys. Lett. A **356** 339-345 (2006).
- [13] Y. Ishimori, Prog. Theor. Phys. **72** 33 (1984).
- [14] R. Myrzakulov, S. Vijayalakshmi, G. Nugmanova, M. Lakshmanan Physics Letters A, **233**, 14-6, 391-396 (1997).
- [15] R. Myrzakulov, S. Vijayalakshmi, R. Syzdykova, M. Lakshmanan, J. Math. Phys., **39**, 2122-2139 (1998).
- [16] R. Myrzakulov, M. Lakshmanan, S. Vijayalakshmi, A. Danlybaeva, J. Math. Phys., **39**, 3765-3771 (1998).
- [17] Myrzakulov R, Danlybaeva A.K, Nugmanova G.N. Theoretical and Mathematical Physics, V.118, 13, P. 441-451 (1999).
- [18] Myrzakulov R., Nugmanova G., Syzdykova R. Journal of Physics A: Mathematical & Theoretical, V.31, 147, P.9535-9545 (1998).
- [19] Myrzakulov R., Daniel M., Amuda R. Physica A., V.234, 13-4, P.715-724 (1997).
- [20] Myrzakulov R., Makhankov V.G., Pashaev O.. Letters in Mathematical Physics, V.16, N1, P.83-92 (1989)
- [21] Myrzakulov R., Makhankov V.G., Makhankov A. Physica Scripta, V.35, N3, P. 233-237 (1987)
- [22] Myrzakulov R., Pashaev O., Kholmurodov Kh. Physica Scripta, V.33, N4, P. 378-384 (1986)
- [23] Anco S.C., Myrzakulov R. Journal of Geometry and Physics, v.60, 1576-1603 (2010)
- [24] Myrzakulov R., Rahimov F.K., Myrzakul K., Serikbaev N.S. *On the geometry of stationary Heisenberg ferromagnets*. In: "Non-linear waves: Classical and Quantum Aspects", Kluwer Academic Publishers, Dordrecht, Netherlands, P. 543-549 (2004)
- [25] Myrzakulov R., Serikbaev N.S., Myrzakul Kur., Rahimov F.K. *On continuous limits of some generalized compressible Heisenberg spin chains*. Journal of NATO Science Series II. Mathematics, Physics and Chemistry, V 153, P. 535-542 (2004)
- [26] Myrzakulov R., Martina L., Kozhamkulov T.A., Myrzakul Kur. *Integrable Heisenberg ferromagnets and soliton geometry of curves and surfaces*. In book: "Nonlinear Physics: Theory and Experiment. II". World Scientific, London, P. 248-253 (2003)
- [27] Myrzakulov R. *Integrability of the Gauss-Codazzi-Mainardi equation in 2+1 dimensions*. In "Mathematical Problems of Nonlinear Dynamics", Proc. of the Int. Conf. "Progress in Nonlinear sciences", Nizhny Novgorod, Russia, July 2-6, 2001, V.1, P.314-319 (2001)
- [28] Chen Chi, Zhou Zi-Xiang. *Darboux Transformation and Exact Solutions of the Myrzakulov-I Equations*. Chin. Phys. Lett., **26**, N8, 080504 (2009)
- [29] Zhao-Wen Yan, Min-Ru Chen, Ke Wu, Wei-Zhong Zhao. J. Phys. Soc. Jpn., **81**, 094006 (2012)
- [30] Yan Zhao-Wen, Chen Min-Ru, Wu Ke, Zhao Wei-Zhong. Commun. Theor. Phys., **58**, 463-468 (2012)
- [31] K.R. Esmakhanova, G.N. Nugmanova, Wei-Zhong Zhao, Ke Wu. *Integrable inhomogeneous Lakshmanan-Myrzakulov equation*, [nlin/0604034]
- [32] Zhen-Huan Zhang, Ming Deng, Wei-Zhong Zhao, Ke Wu. *On the integrable inhomogeneous Myrzakulov-I equation*, [nlin/0603069]
- [33] Martina L, Myrzakul Kur., Myrzakulov R, Soliani G. Journal of Mathematical Physics, V.42, 13, P.1397-1417 (2001).